Yau College Math Competition 2023

Final Probability and Statistics

Individual Exam Problems (June 10-11, 2023)

Choose 3 of the following 4 problems.

Problem 1. Suppose that $Y_n \sim \text{Poisson}(n)$ is a Poisson random variable with parameter $n \in \mathbb{N}^* = \{1, 2, \cdots\}$.

(1) Calculate $\mathrm{E}(Y_n - n)_+$, where $x_+ = \max\{x, 0\}$.

(2) Prove that $\mathrm{E}\frac{(Y_n-n)_+}{\sqrt{n}}$ converges to $\mathrm{E}N_+$ as $n \to \infty$, where $N \sim N(0,1)$ (standard normal).

(3) Use the above results to derive Stirling's formula for the factorial $n!$.

Problem 2. For any random variables ξ and η on the same probability space taking values in $\mathbb{N}^* = \{1, 2, \cdots\}$, assume that $\mathrm{P}(\xi \text{ is divisible by } r) = \mathrm{P}(\eta \text{ is divisible by } r)$, $\forall\ r \in \mathbb{N}^*$. Prove or disprove that $\xi \overset{d}{=} \eta$ (i.e., ξ and η have the same distribution).

Problem 3. Fix some integer $k \geq 2$. For any $n \geq 1$, let $\{X_{n,i} : i = 1, 2, \cdots, k\}$ be i.i.d. uniform random variables taking values in $\{1, 2, \cdots, n\}$. Let $Z_n = \gcd\{X_{n,i} : i = 1, 2, \cdots, k\}$ be the greatest common divisor of $\{X_{n,i} : i = 1, 2, \cdots, k\}$.

(1) Prove that Z_n converges in law to some limit Z with $\mathrm{P}(Z = r) \propto r^{-k}, \forall\ r \in \mathbb{N}^*$.

(2) Choose k numbers independently and uniformly from $\{1, 2, \cdots, n\}$. Let $P_n(k)$ be the probability that these numbers are relatively prime. Find $\lim_{n\to\infty} P_n(k)$.

Problem 4. Let $X_1, \ldots, X_n$ be independent observations. Assume that $X_i \sim \text{Poisson}(i\lambda)$, where λ is an unkonwn parameter.

(1) Find an unbiased estimator of the parameter λ. Is your estimator the best unbiased estimator in terms of minimizing the mean squared error?

(2) Is your estimator asymptotically normal?